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# Nonlinear Operators and Fixed Point Theorems in Hilbert Spaces

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**Abstract.** In this article, we first consider new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from an equilibrium problem in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

## 1 Introduction

Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to H$ is said to be nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We know that if $C$ is a bounded closed convex subset of $H$ and $T : C \to C$ is nonexpansive, then the set $F(T)$ of fixed points of $T$ is nonempty. Further, from Baillon [1] we know the first nonlinear ergodic theorem in a Hilbert space: Let $C$ be a nonempty bounded closed convex subset of $H$ and let $T : C \to C$ be nonexpansive. Then, for any $x \in C$,

$$S_n x = \frac{1}{n} \sum_{k=0}^{n-1} T^k x$$

converges weakly to an element $z \in F(T)$. An important example of nonexpansive mappings in a Hilbert space is a firmly nonexpansive mapping. A mapping $T$ is said to be firmly nonexpansive if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in C;$$

see, for instance, Goebel and Kirk [8]. It is also known that a firmly nonexpansive mapping $T$ can be deduced from an equilibrium problem in a Hilbert space as follows: Let $C$ be a nonempty closed convex subset of $H$ and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying the following conditions:

(A1) $f(x, x) = 0, \quad \forall x \in C$;
(A2) $f$ is monotone, i.e., $f(x, y) + f(y, x) \leq 0, \quad \forall x, y \in C$;
(A3) $\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y), \quad \forall x, y, z \in C$;
(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

We know the following lemma; see, for instance, [2] and [7].

**Lemma 1.1.** *Let $C$ be a nonempty closed convex subset of $H$ and let $f$ be a bifunction from $C \times C$ into $\mathbb{R}$ satisfying (A1), (A2), (A3) and (A4). Then, for any $r > 0$ and $x \in H$, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

*Further, if $T_r x = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C\}$ for all $x \in H$, then the following hold:*

*(1) $T_r$ is single-valued;*
*(2) $T_r$ is firmly nonexpansive, i.e.,*

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle, \quad \forall x, y \in H.$$

Recently, Kohsaka and Takahashi [12] introduced the following nonlinear mapping: Let $E$ be a smooth, strictly convex and reflexive Banach space, let $J$ be the duality mapping of $E$ and let $C$ be a nonempty closed convex subset of $E$. Then, a mapping $S : C \to E$ is said to be nonspreading if

$$\phi(Sx, Sy) + \phi(Sy, Sx) \leq \phi(Sx, y) + \phi(Sy, x), \quad \forall x, y \in C,$$

where $\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2$ for all $x, y \in E$. They considered such a mapping to study the resolvents of a maximal monotone operator in the Banach space. In the case when $E$ is a Hilbert space, we know that $\phi(x,y) = \|x - y\|^2$ for all $x, y \in E$. So, a nonspreading mapping $S$ in a Hilbert space is defined as follows:

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|Sy - x\|^2, \quad \forall x, y \in C.$$

On the other hand, Takahashi [29] found another new nonlinear mapping called a hybrid mapping which is also deduced from a firmly nonexpansive mapping.

In this paper, we first discuss new classes of nonlinear mappings containing the class of firmly nonexpansive mappings which can be deduced from a firmly nonexpansive mapping in a Hilbert space. Further, we deal with fixed point theorems and ergodic theorems for these nonlinear mappings.

## 2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$, respectively. In a Hilbert space, it is known that for all $x, y \in H$ and $\alpha \in \mathbb{R}$,

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2; \tag{2.1}$$

see, for instance, [28]. Further, in a Hilbert space, we have that for all $x, y, z, w \in H$,

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2. \tag{2.2}$$

Let $C$ be a nonempty subset of $H$ and let $T$ be a mapping of $C$ into $H$. We denote by $F(T)$ the set of all fixed points of $T$, that is, $F(T) = \{z \in C : Tz = z\}$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. The following lemma is in [20].

**Lemma 2.1.** *Let $C$ be a nonempty closed convex subset of $H$ and let $f : C \to (-\infty, \infty]$ be a proper convex lower semicontinuous function such that $f(z_m) \to \infty$ as $\|z_m\| \to \infty$. Then there exists an element $z_0 \in C$ such that*

$$f(z_0) = \min\{f(z) : z \in C\}.$$

Let $\mathbb{N}$ be the set of positive integers and let $l^\infty$ be the Banach space of bounded sequences with supremum norm. Let $\mu$ be an element of $(l^\infty)^*$ (the dual space of $l^\infty$). Then, we denote by $\mu(f)$ the value of $\mu$ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional $\mu$ on $l^\infty$ is called a mean if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean $\mu$ is called a Banach limit on $l^\infty$ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on $l^\infty$; see [20] for more details.

# 3 Nonlinear Mappings

Let $H$ be a Hilbert space. Let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into $H$. Then, from [29], we have the following equality:

$$\|Tx - Ty\|^2 = \|x - y - (Tx - Ty)\|^2 - \|x - y\|^2 + 2\langle x - y, Tx - Ty\rangle \tag{3.1}$$

for all $x, y \in C$. We have also from (2.2) that

$$2\langle x - y, Tx - Ty\rangle = \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2. \tag{3.2}$$

Further, we have that

$$\|x - y - (Tx - Ty)\|^2 = \|x - Tx\|^2 + \|y - Ty\|^2 - 2\langle x - Tx, y - Ty\rangle. \tag{3.3}$$

If $T : C \to H$ is firmly nonexpansive, then

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in C.$$

So, we have from (3.1) that for all $x, y \in C$,

$$\begin{aligned} 2\|Tx - Ty\|^2 &\le 2\langle x - y, Tx - Ty\rangle \\ &= \|Tx - Ty\|^2 - \|x - y - (Tx - Ty)\|^2 + \|x - y\|^2 \\ &\le \|Tx - Ty\|^2 + \|x - y\|^2. \end{aligned}$$

Then, we have $\|Tx - Ty\|^2 \le \|x - y\|^2$ and hence $\|Tx - Ty\| \le \|x - y\|$. Such a mapping is nonexpansive. We know that $T : C \to H$ is nonexpansive if and only if

$$2\|Tx - Ty\|^2 \le \|x - Ty\|^2 + \|y - Ts\|^2 - 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C;$$

see [29]. Thus, we can get new classes of nonlinear operators which contain the class of firmly nonexpansive mappings in a Hilbert space. For example, Kohsaka and Takahahi [12] obtained a *nonspreading mapping*. Let $T : C \to H$ be a firmly nonexpansive mapping. Then, we have that for all $x, y \in C$,

$$2\|Tx - Ty\|^2 \le 2\langle x - y, Tx - Ty\rangle.$$

From (3.2), we obtain

$$\begin{aligned}2\|Tx - Ty\|^2 &\leq \|x - Ty\|^2 + \|y - Tx\|^2 - \|x - Tx\|^2 - \|y - Ty\|^2 \\ &\leq \|x - Ty\|^2 + \|y - Tx\|^2.\end{aligned}$$

So, we have

$$2\|Tx - Ty\|^2 \leq \|x - Ty\|^2 + \|y - Tx\|^2.$$

This is a nonspreading mapping. From Iemoto and Takahashi [10], we know the following lemma.

**Lemma 3.1.** *Let $C$ be a nonempty closed convex subset of $H$. Then a mapping $S : C \to H$ is nonspreading if and only if*

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + 2\langle x - Sx, y - Sy\rangle$$

*for all $x, y \in C$.*

Further, from a firmly nonexpansive mapping, i.e.,

$$\|Tx - Ty\|^2 \leq \langle Tx - Ty, x - y\rangle, \quad \forall x, y \in C,$$

we have

$$\|Tx - Ty\|^2 \leq 2\langle Tx - Ty, x - y\rangle, \quad \forall x, y \in C.$$

Such a mapping $T : C \to H$ is called $\frac{1}{2}$-inverse strongly monotone. Takahashi [29] also defined the following *hybrid mapping*, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|y - Tx\|^2 + \|x - Ty\|^2, \quad \forall x, y \in C.$$

From Takahashi [29], we know the following lemma.

**Lemma 3.2.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then a mapping $T : C \to H$ is hybrid if and only if*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C.$$

So, a hybrid mapping $T : C \to H$ is different from a nonspreading mapping. Further, we define a new nonlinear operator from a firmly nonexpansive mapping. We have that for any $x, y \in C$,

$$\begin{aligned}&2\|Tx - Ty\|^2 \leq 2\langle x - y, Tx - Ty\rangle \\ &\iff \|Tx - Ty\|^2 + \|Tx\|^2 + \|Ty\|^2 - 2\langle Tx, Ty\rangle \leq 2\langle x - y, Tx - Ty\rangle \\ &\implies \|Tx - Ty\|^2 - 2\langle Tx, Ty\rangle \leq 2\langle x - y, Tx - Ty\rangle \\ &\iff \|Tx - Ty\|^2 \leq 2\langle Tx, Ty\rangle + 2\langle x - y, Tx - Ty\rangle.\end{aligned}$$

So, we can define a new mapping called a metric mapping, i.e.,

$$\|Tx - Ty\|^2 \leq 2\langle Tx, Ty\rangle + 2\langle x - y, Tx - Ty\rangle, \quad \forall x, y \in C.$$

Let $T : C \to H$ be a nonexpansive mapping and put $A = I - T$. Then, we have from [28] that $A$ is 1/2-inverse strongly monotone, i.e.,

$$\frac{1}{2}\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle, \quad \forall x, y \in C.$$

Let $T : C \to H$ be a nonspreading mapping and put $A = I - T$. Then, we have from Lemma 3.1 and (3.1) that for any $x, y \in C$,

$$\begin{aligned}
\|Ax - Ay\|^2 &= \|x - y - (Ax - Ay)\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\
&= \|Tx - Ty\|^2 - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\
&\leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle - \|x - y\|^2 + 2\langle x - y, Ax - Ay\rangle \\
&= 2\langle Ax, Ay\rangle + 2\langle x - y, Ax - Ay\rangle.
\end{aligned}$$

This implies that $A$ is a metric mapping.

## 4 Generalized Fixed Point Theorem and its Applications

In this section, we obtain a generalized fixed point theorem in a Hilbert space. Before proving the theorem, we can obtain the following lemma from Lemma 2.1.

**Lemma 4.1.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$, let $\{x_n\}$ be a bounded sequence in $H$ and let $\mu$ be a Banach limit. If $g : C \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n\|x_n - z\|^2, \quad \forall z \in C,$$

*then there exists a unique $z_0 \in C$ such that*

$$g(z_0) = \min\{g(z) : z \in C\}.$$

Using Lemma 4.1, we can prove the following generalized fixed point theorem [31].

**Theorem 4.2.** *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into itself. Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded and*

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2, \quad \forall y \in C$$

*for some Banach limit $\mu$. Then, $T$ has a fixed point in $C$.*

*Proof.* Using a Banach limit $\mu$ on $l^\infty$, we can define $g : C \to \mathbb{R}$ as follows:

$$g(z) = \mu_n\|T^n x - z\|^2, \quad \forall z \in C.$$

From Lemma 4.1, there exists a unique $z_0 \in C$ such that

$$g(z_0) = \min\{g(z) : z \in C\}.$$

So, we have

$$g(Tz_0) = \mu_n\|T^n x - Tz_0\|^2 \leq \mu_n\|T^n x - z_0\|^2 = g(z_0).$$

Since $Tz_0$ is in $C$ and $z_0 \in C$ is a unique element such that

$$g(z_0) = \min\{g(z) : z \in C\},$$

we have $Tz_0 = z_0$. This completes the proof. □

Using Theorem 4.2, we can obtain some fixed point theorems. The following is the well-known fixed point theorem for nonexpansive mappings in a Hilbert space; see, for instance, [28].

**Theorem 4.3.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a nonexpansive mapping, i.e.,*

$$\|Tx - Ty\| \leq \|x - y\|, \quad \forall x, y \in C.$$

*Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.*

*Proof.* Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2.$$

So, we have

$$\mu_n \|T^n x - Ty\|^2 = \mu_n \|T^{n+1}x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2$$

for all $y \in C$. By Theorem 4.2, $T$ has a fixed point in $C$. □

The following is a fixed point theorem for nonspreading mappings in a Hilbert space.

**Theorem 4.4** ([12]). *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a nonspreading mapping, i.e.,*

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

*Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.*

*Proof.* Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \leq \|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n \|T^n x - Ty\|^2 &= 2\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq \mu_n \|T^{n+1}x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \\ &= \mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, $T$ has a fixed point in $C$. □

The following is a fixed point theorem for hybrid mappings in a Hilbert space.

**Theorem 4.5** ([29]). *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a hybrid mapping, i.e.,*

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \langle x - Tx, y - Ty \rangle, \quad \forall x, y \in C.$$

*Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.*

*Proof.* Let $\mu$ be a Banach limit on $l^\infty$. We know from Lemma 3.2 that a mapping $T : C \to C$ is hybrid if and only if

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

So, for any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2 + \|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$\begin{aligned} 3\mu_n \|T^n x - Ty\|^2 &= 3\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq 2\mu_n \|T^n x - y\|^2 + \mu_n \|T^n x - Ty\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, $T$ has a fixed point in $C$. □

We can also prove the following two fixed point theorems in a Hilbert space.

**Theorem 4.6.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

*Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.*

*Proof.* Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$2\|T^{n+1}x - Ty\|^2 \leq \|T^n x - y\|^2 + \|T^{n+1}x - y\|^2.$$

So, we have

$$\begin{aligned} 2\mu_n \|T^n x - Ty\|^2 &= 2\mu_n \|T^{n+1}x - Ty\|^2 \\ &\leq 2\mu_n \|T^n x - y\|^2 \end{aligned}$$

and hence

$$\mu_n \|T^n x - Ty\|^2 \leq \mu_n \|T^n x - y\|^2.$$

By Theorem 4.2, $T$ has a fixed point in $C$. □

**Theorem 4.7.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a mapping such that*

$$3\|Tx - Ty\|^2 \leq 2\|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

*Suppose that there exists an element $x \in C$ such that $\{T^n x\}$ is bounded. Then, $T$ has a fixed point in $C$.*

*Proof.* Let $\mu$ be a Banach limit on $l^\infty$. For any $n \in \mathbb{N}$ and $y \in C$, we have

$$3\|T^{n+1}x - Ty\|^2 \leq 2\|T^{n+1}x - y\|^2 + \|T^n x - Ty\|^2.$$

So, we have

$$3\mu_n\|T^n x - Ty\|^2 \leq 2\mu_n\|T^n x - y\|^2 + \mu_n\|T^n x - Ty\|^2$$

and hence

$$\mu_n\|T^n x - Ty\|^2 \leq \mu_n\|T^n x - y\|^2.$$

By Theorem 4.2, $T$ has a fixed point in $C$. □

We also know the following theorem by Ray [15].

**Theorem 4.8.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:*

*(i) Every nonexpansive mapping of $C$ into itself has a fixed point in $C$;*
*(ii) $C$ is bounded.*

Using Ray's theorem, we can prove the following theorem [29].

**Theorem 4.9.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:*

*(i) Every hybrid mapping of $C$ into itself has a fixed point in $C$;*
*(ii) $C$ is bounded.*

*Proof.* From Theorem 4.5, we know that (ii) implies (i). Let us show that (i) implies (ii). We know that every firmly nonexpansive mapping is a hybrid mapping. So, the class of hybrid mappings of $C$ into itself contains the class of firmly nonexpansive mappings of $C$ into itself. To show (i) $\Longrightarrow$ (ii), it is sufficient to show that if every firmly nonexpansive mapping in $C$ into itself has a fixed point in $C$, then every nonexpansive mapping of $C$ into itself has a fixed point in $C$. Let $T$ be a nonexpansive mapping of $C$ into itself. Then, $S = \frac{1}{2}I + \frac{1}{2}T$ is a firmly nonexpansive mapping. Further, it is not difficult to show $F(T) = F(S)$. So, every firmly nonexpansive mapping in $C$ into itself has a fixed point in $C$ if and only if every nonexpansive mapping of $C$ into itself has a fixed point in $C$. This completes the proof. □

Similarly, we have the following theorem.

**Theorem 4.10.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Then, the following are equivalent:*

*(i) Every nonspreading mapping of $C$ into itself has a fixed point in $C$;*
*(ii) $C$ is bounded.*

## 5 Nonlinear Erdodic Theorems

Baillon [1] proved the first nonlinear ergodic theorem for nonexpansive mappings in a Hilbert space.

**Theorem 5.1.** *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a nonexpansive mapping of $C$ into itself such that $F(T)$ is nonempty. Then, for any $x \in C$,*

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

*converges weakly to an element $z \in F(T)$.*

We can also prove the following nonlinear ergodic theorem [31] for our nonlinear operators in a Hilbert space. Before proving it, we need, for example, the following result [31].

**Lemma 5.2.** *Let $H$ be a Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T : C \to C$ be a mapping such that*

$$2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C.$$

*Then $T$ is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ imply $u \in F(T)$.*

*Proof.* Let $\{x_n\} \subset C$ be a sequence such that $x_n \rightharpoonup u$ and $x_n - Tx_n \to 0$ as $n \to \infty$. Then the sequences $\{x_n\}$ and $\{Tx_n\}$ are bounded. Suppose that $u \neq Tu$. From Opial's theorem [14], we have

$$\begin{aligned}
\liminf_{n\to\infty} \|x_n - u\|^2 &< \liminf_{n\to\infty} \|x_n - Tu\|^2 \\
&= \liminf_{n\to\infty} \|x_n - Tx_n + Tx_n - Tu\|^2 \\
&= \liminf_{n\to\infty} \|Tx_n - Tu\|^2 \\
&\le \liminf_{n\to\infty} \frac{1}{2}(\|x_n - u\|^2 + \|Tx_n - u\|^2) \\
&= \liminf_{n\to\infty} \frac{1}{2}(\|x_n - u\|^2 + \|Tx_n - x_n + x_n - u\|^2) \\
&= \liminf_{n\to\infty} \|x_n - u\|^2 .
\end{aligned}$$

This is a contradiction. Hence we get the conclusion. □

**Theorem 5.3.** *Let $H$ be a Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $T$ be a mapping of $C$ into itself such that $F(T)$ is nonempty. Suppose that $T$ satisfies one of the following conditions:*

*(i)* *$T$ is nonspreading;*
*(ii)* *$T$ is hybrid;*
*(iii)* $2\|Tx - Ty\|^2 \le \|x - y\|^2 + \|Tx - y\|^2, \quad \forall x, y \in C;$
*(iv)* $3\|Tx - Ty\|^2 \le 2\|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$

*Then, for any* $x \in C$,

$$S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$$

*converges weakly to an element* $z \in F(T)$.

*Proof.* Let us prove the case of (iii) by using Lemma 5.2. We first show that $F(T)$ is closed and convex. It follows from Lemma 5.2 that $F(T)$ is closed. In fact, let $\{x_n\} \subset F(T)$ and $x_n \to z$. Then, we have $x_n \rightharpoonup z$ and $x_n - Tx_n = 0$. So, from Lemma 5.2 we have $z = Tz$. This implies that $F(T)$ is closed. Let us show that $F(T)$ is convex. Let $x, y \in F(T)$ and let $\alpha \in [0,1]$. Put $z = \alpha x + (1-\alpha)y$. Then, we have from (2.1) that

$$\begin{aligned}
\|z - Tz\|^2 &= \|\alpha x + (1-\alpha)y - Tz\|^2 \\
&= \alpha\|x - Tz\|^2 + (1-\alpha)\|y - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha\|Tx - Tz\|^2 + (1-\alpha)\|Ty - Tz\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&\le \alpha(1-\alpha)^2\|x - y\|^2 + (1-\alpha)\alpha^2\|x - y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \\
&= \alpha(1-\alpha)(1 - \alpha + \alpha - 1)\|x - y\|^2 \\
&= 0.
\end{aligned}$$

So, we have $Tz = z$. This implies that $F(T)$ is convex. Let $x \in C$ and let $P$ be the metric projection of $H$ onto $F(T)$. Then, we have

$$\begin{aligned}
\|PT^n x - T^n x\| &\le \|PT^{n-1}x - T^n x\| \\
&= \|TPT^{n-1}x - T^n x\| \\
&\le \|PT^{n-1}x - T^{n-1}x\|.
\end{aligned}$$

This implies that $\{\|PT^n x - T^n x\|\}$ is nonincreasing. We also know that for any $v \in C$ and $u \in F(T)$,

$$\langle v - Pv, Pv - u\rangle \ge 0$$

and hence

$$\|v - Pv\|^2 \le \langle v - Pv, v - u\rangle.$$

So, we get

$$\begin{aligned}
\|Pv - u\|^2 &= \|Pv - v + v - u\|^2 \\
&= \|Pv - v\|^2 - 2\langle Pv - v, u - v\rangle + \|v - u\|^2 \\
&\le \|v - u\|^2 - \|Pv - v\|^2.
\end{aligned}$$

Let $m, n \in \mathbb{N}$ with $m \ge n$. Putting $v = T^m x$ and $u = PT^n x$, we have

$$\begin{aligned}
\|PT^m x - PT^n x\|^2 &\le \|T^m x - PT^n x\|^2 - \|PT^m x - T^m x\|^2 \\
&\le \|T^n x - PT^n x\|^2 - \|PT^m x - T^m x\|^2.
\end{aligned}$$

So, $\{PT^n x\}$ is a Cauchy sequence. Since $F(T)$ is closed, $\{PT^n x\}$ converges strongly to an element $p$ of $F(T)$. Take $u \in F(T)$. Then we obtain that for any $n \in \mathbb{N}$,

$$\|S_n x - u\| \le \frac{1}{n}\sum_{k=0}^{n-1}\|T^k x - u\| \le \|x - u\|.$$

So, $\{S_n x\}$ is bounded and hence there exists a weakly convergent subsequence $\{S_{n_i} x\}$ of $\{S_n x\}$. If $S_{n_i} x \rightharpoonup v$, then we have $v \in F(T)$. In fact, for any $y \in C$ and $k \in \mathbb{N} \cup \{0\}$, we have that

$$\begin{aligned}
2\|T^{k+1}x - Ty\|^2 &\le \|T^k x - y\|^2 + \|T^{k+1}x - y\|^2 \\
&= \|T^k x - Ty\|^2 + 2\langle T^k x - Ty, Ty - y\rangle + \|Ty - y\|^2 \\
&+ \|T^{k+1}x - Ty\|^2 + 2\langle T^{k+1}x - Ty, Ty - y\rangle + \|Ty - y\|^2.
\end{aligned}$$

So, we obtain that

$$\begin{aligned}
\|T^{k+1}x - Ty\|^2 \le \|T^k x - Ty\|^2 &+ 2\langle T^k x - Ty, Ty - y\rangle \\
&+ 2\langle T^{k+1}x - Ty, Ty - y\rangle + 2\|Ty - y\|^2.
\end{aligned}$$

Summing these inequalities with respect to $k = 0, 1, \dots, n-1$, we have

$$\begin{aligned}
\|T^n x - Ty\|^2 &\le \|x - Ty\|^2 + 2\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y\rangle \\
&+ 2\langle \sum_{k=0}^{n-1} T^{k+1}x - nTy, Ty - y\rangle + 2n\|Ty - y\|^2 \\
&= \|x - Ty\|^2 + 4\langle \sum_{k=0}^{n-1} T^k x - nTy, Ty - y\rangle \\
&+ 2\langle T^n x - x, Ty - y\rangle + 2n\|Ty - y\|^2.
\end{aligned}$$

Deviding this inequality by $n$, we have

$$\begin{aligned}
\frac{1}{n}\|T^n x - Ty\|^2 \le \frac{1}{n}\|x - Ty\|^2 &+ 4\langle S_n x - Ty, Ty - y\rangle \\
&+ \frac{2}{n}\langle T^n x - x, Ty - y\rangle + 2\|Ty - y\|^2,
\end{aligned}$$

where $S_n x = \frac{1}{n}\sum_{k=0}^{n-1} T^k x$. Replacing $n$ by $n_i$ and letting $n_i \to \infty$, we obtain from $S_{n_i} x \rightharpoonup v$ that

$$0 \le 2\|Ty - y\|^2 + 4\langle v - Ty, Ty - y\rangle.$$

Putting $y = v$, we have $0 \le 2\|Tv - v\|^2 + 4\langle v - Tv, Tv - v\rangle$ and hence

$$0 \le \|Tv - v\|^2 + 2\langle v - Tv, Tv - v\rangle.$$

So, we have $0 \le -\|Tv - v\|^2$ and hence $Tv = v$. To complete the proof of (iii), it is sufficient to show that if $S_{n_i} x \rightharpoonup v$ and $p = \lim_{n\to\infty} PT^n x$, then $v = p$. We have that for any $u \in F(T)$,

$$\langle T^k x - PT^k x, PT^k x - u\rangle \ge 0.$$

Since $\{\|T^k x - PT^k x\|\}$ is nonincreasing, we have

$$\begin{aligned}
\langle u - p, T^k x - PT^k x\rangle &\le \langle PT^k x - p, T^k x - PT^k x\rangle \\
&\le \|PT^k x - p\| \cdot \|T^k x - PT^k x\| \\
&\le \|PT^k x - p\| \cdot \|x - Px\|.
\end{aligned}$$

Adding these inequalities from $k = 0$ to $k = n - 1$ and dividing $n$, we have

$$\langle u - p, S_n x - \frac{1}{n}\sum_{k=0}^{n-1} PT^k x\rangle \leq \frac{\|x - Px\|}{n}\sum_{k=0}^{n-1} \|PT^k x - p\|.$$

Since $S_{n_i} x \rightharpoonup v$ and $PT^k x \to p$, we have

$$\langle u - p, v - p\rangle \leq 0.$$

We know $v \in F(T)$. So, putting $u = v$, we have $\langle v - p, v - p\rangle \leq 0$ and hence $\|v - p\|^2 \leq 0$. So, we obtain $v = p$. This completes the proof of (iii). See [31] for the proofs of (1), (ii) and (iv). □

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